

CAN THE CABIBBO MIXING ORIGINATE FROM NONCOMMUTATIVE EXTRA DIMENSIONS?

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Abstract. Treating hadronic flavor symmetries with quantum algebras $U_q(su_n)$ leads to interesting consequences such as: new mass sum rules for hadrons 1^- , $\frac{1}{2}^+$, $\frac{3}{2}^+$ of improved accuracy; possibility to label different flavors topologically - by torus winding number; properly fixed deformation parameter q in case of baryons is linked in a simplest way to the Cabibbo angle θ_C , that suggests for θ_C the exact value $\frac{\pi}{14}$. In this connection, we discuss the possibility that this angle and the Cabibbo mixing as a whole take its origin in noncommutativity of some additional, with regard to 3+1, space-time dimensions.

1. Introduction

The problem of fermion flavors, mixings and masses (see e.g., [1]) belongs to most puzzling ones in particle physics. The Cabibbo mixing first introduced for three lightest flavors in the context of weak decays [2] involves the angle θ_C . Importance of this concept was further confirmed after its generalization to mixing of 3 families [3]. Due to Wolfenstein parametrization [4] of CKM matrix, the Cabibbo angle now plays a prominent role: not only CKM matrix elements V_{ij} , but also the quark (and even lepton) mass ratios are often expressed as powers of small parameter $\lambda = \sin \theta_C \approx 0.22$. No doubt, it is necessary to know the value of λ as precise as possible. In this respect, the main bonus of our approach to flavor symmetries, based on quantum algebras, is that it suggests *theoretically motivated* exact value for θ_C , namely, $\theta_C = \frac{\pi}{14}$. As further implication, it leads us to a conjecture of possible noncommutative-geometric origin of the Cabibbo mixing, and our aim here is to argue this may indeed be the case. Below, when treating baryon masses, we restrict ourselves with 4 flavors including u-, d-, s-, and c- quarks. Basic tool of the approach used is the representation theory of

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quantum algebras [5] $U_q(su_n)$ adopted, instead of conventional $SU(n)$, to describe flavor symmetries classifying hadrons into multiplets.

2. Vector meson masses: q -deformation replaces (singlet) mixing

We use¹ Gelfand-Tsetlin basis vectors for meson states from (n^2-1) -plet of ' n -flavor' $U_q(u_n)$ embedded into $\{(n+1)^2-1\}$ -plet of 'dynamical' $U_q(u_{n+1})$; construct mass operator \hat{M}_n invariant under the 'isospin+hypercharge' q -algebra $U_q(u_2)$ from generators of dynamical algebra $U_q(u_{n+1})$ (e.g., $\hat{M}_3 = M_0 \mathbf{1} + \gamma_3 A_{34} A_{43} + \delta_3 A_{43} A_{34}$); calculate the expressions for masses $m_{V_i} \equiv \langle V_i | \hat{M}_3 | V_i \rangle$ - these involve M_0 , symmetry breaking parameters γ_3, δ_3 , and the q -parameter. In particular, for $n = 3$ we obtain

$$m_\rho = M_0, \quad m_{K^*} = M_0 - \gamma_3, \quad m_{\omega_8} = M_0 - 2 \frac{[2]_q}{[3]_q} \gamma_3, \quad (1)$$

where $[x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}}$ is the q -number that 'deforms' a number x and, to have equal masses for particles and their anti's, $\delta_3 = \gamma_3$ was set. q -Dependence appears only in the mass of ω_8 (isosinglet in $U_q(su_3)$ -octet). Excluding M_0, γ_3 , the q -analog of Gell-Mann - Okubo (GMO) relation is [8] :

$$m_{\omega_8} + \left(2 \frac{[2]_q}{[3]_q} - 1\right) m_\rho = 2 \frac{[2]_q}{[3]_q} m_{K^*}. \quad (2)$$

In the limit $q = 1$ (then, $\frac{[2]_q}{[3]_q} = \frac{2}{3}$), this reduces to usual GMO formula $3m_{\omega_8} + m_\rho = 4m_{K^*}$ which needs singlet mixing [9]. However, it also yields

$$m_{\omega_8} + m_\rho = 2m_{K^*} \quad \text{if} \quad q = e^{i\pi/5} \quad (\text{then, } [2]_q = [3]_q). \quad (3)$$

With $m_{\omega_8} \equiv m_\phi$, and no mixing, eq.(3) coincides with nonet mass formula of Okubo [10] agreeing *ideally* with data [11].

For $3 < n \leq 6$ mass operator is constructed analogously. Again, calculations show: only singlets $\omega_{15}, \omega_{24}, \omega_{35}$ of (n^2-1) -plets of $U_q(u_n)$ contain q -dependence. As result, we get the q -deformed mass relations [8, 6, 7]:

$$[n]_{(q)} m_{\omega_{n^2-1}} + (b_{n;q} + 2n - 4) m_\rho = 2 m_{D_n^*} + (c_{n;q} + 2) \sum_{r=3}^{n-1} m_{D_r^*}, \quad (4)$$

$$b_{n;q} \equiv n c_{n;q} - 6 [n]_{(q)}^2 + \left(\frac{24}{[2]_q} - 1\right) [n]_{(q)}, \quad c_{n;q} \equiv 2 [n]_{(q)}^2 - \frac{8}{[2]_q} [n]_{(q)},$$

where $[n]_{(q)} \equiv [n]_q / [n-1]_q$. Then, natural fixation by setting $[n]_q = [n-1]_q$,

¹ For more details concerning this approach see refs. [6, 7, 13].

$n = 4, 5, 6$, leads to the higher analogs of Okubo's nonet sum rule:

$$m_{\omega_{15}} + (5 - 8/[2]_{q_4})m_\rho = 2 m_{D^*} + (4 - 8/[2]_{q_4})m_{K^*} \quad (5)$$

$$m_{\omega_{24}} + (9 - 16/[2]_{q_5})m_\rho = 2 m_{D_b^*} + (4 - 8/[2]_{q_5})(m_{D^*} + m_{K^*}) \quad (6)$$

$$m_{\omega_{35}} + (13 - 24/[2]_{q_6})m_\rho = 2 m_{D_t^*} + (4 - 8/[2]_{q_6})(m_{D_b^*} + m_{D^*} + m_{K^*}). \quad (7)$$

Here $q_n = e^{i\pi/(2n-1)}$ are the values that solve eqns. $[n]_q - [n-1]_q = 0$. Like in the case with $m_{\omega_8} \equiv m_\phi$, it is meant in (5)-(7) that J/ψ is put in place of ω_{15} , Υ in place of ω_{24} , toponium in place of ω_{35} (i.e., no mixing!).

The q -polynomials $[n]_q - [n-1]_q$ have a topological meaning.

3. Torus knots and topological labelling of flavors

Polynomials $[n]_q - [n-1]_q \equiv P_n(q)$, by their roots, reduce q -analogs (2), (4) to realistic mass sum rules (MSR) (3), (5)-(7). And, due to property (i) $P_n(q) = P_n(q^{-1})$, (ii) $P_n(1) = 1$, they coincide [8, 7] with such knot invariants as Alexander polynomials $\Delta(q)\{(2n-1)_1\}$ of $(2n-1)_1$ -torus knots. E.g.,

$$[3]_q - [2]_q = q^2 + q^{-2} - q - q^{-1} + 1 \equiv \Delta(q)\{5_1\},$$

$$[4]_q - [3]_q = q^3 + q^{-3} - q^2 - q^{-2} + q + q^{-1} - 1 \equiv \Delta(q)\{7_1\}$$

correspond to the 5_1 - and 7_1 -knots. Since the q -deuce in (4) can be linked to the trefoil (or 3_1 -) knot: $[2]_q - 1 = q + q^{-1} - 1 \equiv \Delta(q)\{3_1\}$, *all the q -dependence* in masses of ω_{n^2-1} and in coefficients in (2),(4) is expressible through Alexander polynomials. Namely, $\frac{[3]_q}{[2]_q} = 1 + \frac{\Delta\{5_1\}}{\Delta\{3_1\}+1}$,

$$\frac{[n]_q}{[n-1]_q} = 1 + \frac{\Delta\{(2n-1)_1\}}{[n-1]_q} = 1 + \frac{\Delta\{(2n-1)_1\}}{1 + \sum_{r=2}^{n-1} \Delta\{(2r-1)_1\}}, \quad n = 4, 5, 6. \quad (8)$$

The values q_n are thus roots of respective Alexander polynomials. For each n , the 'senior' (numerator) polynomial in $\frac{[3]_q}{[2]_q}$ and (8) is specified: by its root, it 'singles out' the corresponding MSR from q -deformed analog.

Thus, the q -parameter for each n is fixed in a rigid way as a root q_n of $\Delta\{(2n-1)_1\}$, contrary to the choice of q by fitting in other phenomenological applications [12]. Moreover, using flavor q -algebras along with 'dynamical' q -algebras according to $U_q(u_n) \subset U_q(u_{n+1})$, we gain: the torus knots 5_1 , 7_1 , 9_1 , 11_1 are put into correspondence [6, 7] with vector quarkonia $s\bar{s}$, $c\bar{c}$, $b\bar{b}$, and $t\bar{t}$ respectively. In a sense, the polynomial $P_n(q) \equiv [n]_q - [n-1]_q$ by its root $q(n)$ determines the value of q (deformation strength) for each n and thus serves as *defining polynomial* for

the MSR/quarkonium/ flavor corresponding to n . Hence, the applying of q -algebras suggests a possibility of *topological labeling of flavors*: fixed number n corresponds to $2n-1$ overcrossings of 2-strand braids whose closure gives these $(2n-1)_1$ -torus knots. With the form $(2n-1, 2)$ of same torus knots this means the correspondence $n \leftrightarrow w \equiv 2n-1$, w being the winding number around tube of torus (winding number around hole is 2).

4. Defining q -polynomials for octet baryon mass sum rules

Analogous scheme was applied to baryons $\frac{1}{2}^+$ too. Excluding undetermined constants M_0, α, β from final obtained expressions for $M_N, M_\Xi, M_\Lambda, M_\Sigma$ leads to the q -deformed mass relations (MRs) of the form [6, 7, 13]

$$[2]M_N + \frac{[2]}{[2]-1}M_\Xi = [3]M_\Lambda + \left(\frac{[2]^2}{[2]-1} - [3] \right) M_\Sigma + \frac{A_q}{B_q} (M_\Xi + [2]M_N - [2]M_\Sigma - M_\Lambda) \quad (9)$$

where A_q and B_q are certain polynomials of $[2]_q$ with non-overlapping sets of zeros. It is important that different dynamical representations produce differing pairs A_q, B_q . Any A_q possesses the factor $([2]_q - 2)$ and thus the 'classical' zero $q = 1$. In the limit $q = 1$ each q -deformed mass relation reduces to the standard GMO sum rule $M_N + M_\Xi = \frac{1}{2}M_\Sigma + \frac{3}{2}M_\Lambda$ for octet baryons (its accuracy is 0.58%). At some values of q which are zeros of particular A_q other than $q = 1$, we obtain MSRs which hold with better accuracy than the GMO one. The two new MSRs

$$q = e^{i\pi/6} \Rightarrow M_N + \frac{1+\sqrt{3}}{2}M_\Xi = \frac{2}{\sqrt{3}}M_\Lambda + \frac{9-\sqrt{3}}{6}M_\Sigma \quad (0.22\%) \quad (10)$$

$$q = e^{i\pi/7} \Rightarrow M_N + \frac{1}{[2]_{q7}-1}M_\Xi = \frac{1}{[2]_{q7}-1}M_\Lambda + M_\Sigma \quad (0.07\%) \quad (11)$$

result [6,7,13] from two different dynamical representations $D^{(1)}$ and $D^{(2)}$ whose respective polynomials $A_q^{(1)}$ and $A_q^{(2)}$ possess zeros $q = e^{i\pi/6}$ and $q = e^{i\pi/7}$. The choice with $q = e^{i\pi/7}$ turns out to be the best possible one.²

The sum rule (10) was first derived [6] from a specific dynamical representation (irrep) $D^{(1)}$ of $U_q(u_{4,1})$. However, the 'compact' dynamical $U_q(u_5)$ is equally well suited. Among the admissible dynamical irreps there exist an entire series of irreps (numbered by integer m , $6 \leq m < \infty$) which produce the corresponding infinite set of MSRs:

$$M_N + \frac{1}{[2]_{q_m}-1}M_\Xi = \frac{[3]_{q_m}}{[2]_{q_m}}M_\Lambda + \left(\frac{[2]_{q_m}}{[2]_{q_m}-1} - \frac{[3]_{q_m}}{[2]_{q_m}} \right) M_\Sigma \quad (12)$$

² In sec. 8 we argue that this value of q is linked to the Cabibbo angle: $\theta_8 \equiv \frac{\pi}{7} = 2\theta_C$.

with $q_m = e^{i\pi/m}$. Each of these shows better agreement with data than the classical GMO one. Few of them, including the MSRs (10), (11) and the 'classical' GMO which corresponds to $q_\infty = 1$, are shown in the table.

$\theta = \frac{\pi}{m}$	(RHS-LHS), MeV	$\frac{ \text{RHS-LHS} }{\text{RHS}}, \%$
π/∞	26.2	0.58
$\pi/30$	25.42	0.56
$\pi/12$	20.2	0.44
$\pi/8$	10.39	0.23
$\pi/7$	3.26	0.07
$\pi/6$	-10.47	0.22

Comparing (12) with (9) shows that the vanishing of $\frac{A_q}{B_q}$ is crucial for obtaining this discrete set of MSRs and for providing a kind of 'discrete fitting'. Thus, A_q serves as *defining* polynomial for the corresponding MSR.

Since $[2]_{q_7} = q_7 + \frac{1}{q_7} = 2 \cos \frac{\pi}{7}$, the MSR (11) takes the equivalent form

$$M_\Xi - M_N + M_\Sigma - M_\Lambda = (2 \cos \frac{\pi}{7})(M_\Sigma - M_N) \quad (13)$$

which exhibits some similarity with decuplet mass formula given below.

5. Decuplet baryons: universal q -deformed mass relation

In the case of $SU(3)$ -decuplet baryons $\frac{3}{2}^+$, the conventional 1st order symmetry breaking yields [9] equal spacing rule (ESR) for isoplet members in **10**-plet. Empirical data show for $M_{\Sigma^*} - M_\Delta$, $M_{\Xi^*} - M_{\Sigma^*}$ and $M_\Omega - M_{\Xi^*}$ noticeable deviation from ESR: $152.6 \text{ MeV} \leftrightarrow 148.8 \text{ MeV} \leftrightarrow 139.0 \text{ MeV}$. Use of the q -algebras $U_q(su_n)$ instead of $SU(n)$ provides natural improvement. From evaluations of decuplet masses in two distinct particular irreps of the dynamical algebra $U_q(u_{4,1})$, the q -deformed mass relation

$$(1/[2]_q)(M_{\Sigma^*} - M_\Delta + M_\Omega - M_{\Xi^*}) = M_{\Xi^*} - M_{\Sigma^*}, \quad [2]_q \equiv q + q^{-1}, \quad (14)$$

was derived [14]. As proven there, this mass relation is universal - it results from each admissible irrep (which contains $U_q(su_3)$ -decuplet embedded in 20 -plet of $U_q(su_4)$) of the dynamical $U_q(u_{4,1})$. With empirical masses [11], the formula (14) is successful if $[2]_q \simeq 1.96$. *Pure phase* $q = e^{i\theta}$ (or $[2]_q = 2 \cos \theta$) with $\theta = \theta_{10} \simeq \frac{\pi}{14}$ provides excellent agreement with data (below, we argue that $\theta_{10} = \theta_C$). Notice a similarity of eq.(14) with the MR

$$(1/2)(M_{\Sigma^*} - M_\Delta + M_\Omega - M_{\Xi^*}) = M_{\Xi^*} - M_{\Sigma^*} \quad (15)$$

obtained earlier in diverse contexts [15]: by tensor method, in additive quark model with general pair interaction, in a diquark–quark model, in modern chiral perturbation theory. Such model-independence of (15) stems because each of these approaches accounts 1st and 2nd order of $SU(3)$ -breaking.

The q -deformed MSR (14) is universal even in a wider sense: it results from admissible irreps (containing $U_q(su_4)$ 20-plet) of both $U_q(su_{4,1})$ and the 'compact' dynamical $U_q(su_5)$. Say, within a dynamical irrep {4000} of $U_q(su_5)$ calculation yields: $M_\Delta = M_{\mathbf{10}} + \beta$, $M_{\Sigma^*} = M_{\mathbf{10}} + [2]\beta + \alpha$, $M_{\Xi^*} = M_{\mathbf{10}} + [3]\beta + [2]\alpha$, $M_\Omega = M_{\mathbf{10}} + [4]\beta + [3]\alpha$, from which (14) stems. On the other hand, these four masses can be comprised by single formula

$$M_{D_i} = M(Y(D_i)) = M_{\mathbf{10}} + \alpha[1 - Y(D_i)] + \beta[2 - Y(D_i)] \quad (16)$$

with explicit dependence on Y (hypercharge). If $q = 1$, this reduces to $M_{D_i} = \tilde{M}_{\mathbf{10}} + a Y(D_i)$, i.e., *linear dependence on hypercharge Y* (or strangeness) where $a = -\alpha - \beta$, $\tilde{M}_{\mathbf{10}} = M_{\mathbf{10}} + \alpha + 2\beta$.

6. Nonpolynomial $SU(3)$ -breaking effects in baryon masses

Formula (16) *involves highly nonlinear dependence* of mass on hypercharge (it is Y that causes $SU(3)$ -breaking for decuplet). Since for q -number $[N]$ we have $[N] = q^{N-1} + q^{N-3} + \dots + q^{-N+3} + q^{-N+1}$ (N terms) this shows exponential Y -dependence of masses. Such high nonlinearity makes (14) and (16) radically different from the abovementioned result (15) of traditional treatment that accounts for effects linear and quadratic in Y .

For octet baryon masses, high nonlinearity (*nonpolynomiality*) in $SU(3)$ -breaking effectively accounted by the model was demonstrated in [13]. For this, the expressions for (isoplet members of) octet masses with explicit dependence on hypercharge Y and isospin I , through $I(I+1)$, are used. The typical matrix element (μ_1, μ_2 are functions of irrep labels m_{15}, m_{55}):

$$\begin{aligned} \langle B_i | A_{34} A_{45} A_{54} A_{43} | B_i \rangle &= [2]^{-1} [3]^{-1} \left([Y/2][Y/2+1] - [I][I+1] \right) \mu_1(m_{15}, m_{55}) \\ &\quad - [2]^{-1} [5]^{-1} \left([Y/2-1][Y/2-2] - [I][I+1] \right) \mu_2(m_{15}, m_{55}), \end{aligned}$$

contributing to octet baryon masses, illustrates the dependence. From definition of q -bracket $[n] = \frac{\sin(nh)}{\sin(h)}$, $q = \exp(ih)$, it is clearly seen that baryon masses depend on hypercharge Y and isospin I (hence, on $SU(3)$ -breaking effects) in highly nonlinear - *nonpolynomial* - fashion.

The ability to take into account highly nontrivial symmetry breaking effects by applying q -analogs $U_q(su_n)$ of usual flavor symmetries is much alike the fact demonstrated in [16] that, by exploiting appropriate

free q -deformed structure one is able to efficiently study the properties of (undeformed) quantum-mechanical systems with complicated interactions.

7. To use or not to use the Hopf-algebra structure

An alternative, as regards (9), version of q -deformed analog can be derived [13] using for the symmetry breaking part of mass operator a component of q -tensor operator - this clearly implies [17] the Hopf algebra structure (comultiplication, antipode) of the $U_q(su_n)$ quantum algebras. Let us briefly discuss such version. We use q -tensor operators (V_1, V_2, V_3) resp. $(V_{\bar{1}}, V_{\bar{2}}, V_{\bar{3}})$ formed from elements of $U_q(su_4)$ and transforming as $\mathbf{3}$ resp. $\mathbf{3}^*$ under the adjoint action of $U_q(su_3)$. With H_1, H_2 as Cartan elements and with notation $[X, Y]_q \equiv XY - qYX$, the components (V_1, V_2, V_3) read

$$V_1 = [E_1^+, [E_2^+, E_3^+]_q]_q q^{-H_1/3 - H_2/6}, \quad V_2 = [E_2^+, E_3^+]_q q^{H_1/6 - H_2/6},$$

$$V_3 = E_3^+ q^{H_1/6 + H_2/3}, \quad (17)$$

and similarly for $(V_{\bar{1}}, V_{\bar{2}}, V_{\bar{3}})$ (see [13]), of which we here only give

$$V_{\bar{3}} = q^{H_1/6 + H_2/3} E_3^-. \quad (18)$$

Clearly, $U_q(su_3)$ is broken to $U_q(su_2)$. Like in the nondeformed case of $su(3)$ broken to its isospin subalgebra $su(2)$, the form of mass operator is

$$\hat{M} = \hat{M}_0 + \hat{M}_8 \quad (19)$$

where \hat{M}_0 is $U_q(su_3)$ -invariant and \hat{M}_8 transforms as $I=0, Y=0$ component of tensor operator of $\mathbf{8}$ -irrep of $U_q(su_3)$. If $|B_i\rangle$ is a basis vector of carrier space of $\mathbf{8}$ which corresponds to some baryon B_i , the mass of B_i is given by $M_{B_i} = \langle B_i | \hat{M} | B_i \rangle$. The irrep $\mathbf{8}$ occurs twice in the decomposition of $\mathbf{8} \otimes \mathbf{8}$. This, and the Wigner-Eckart theorem for $U_q(su_n)$ [18] applied to q -tensor operators under irrep $\mathbf{8}$ of $U_q(su_3)$, lead to the mass operator of the form $\hat{M} = M_0 \mathbf{1} + \alpha V_8^{(1)} + \beta V_8^{(2)}$ and thus to

$$M_{B_i} = \langle B_i | (M_0 \mathbf{1} + \alpha V_8^{(1)} + \beta V_8^{(2)}) | B_i \rangle \quad (20)$$

where $V_8^{(1)}$ and $V_8^{(2)}$ are two distinct tensor operators which both transform as $I=0, Y=0$ component of irrep $\mathbf{8}$ of $U_q(su_3)$; M_0, α, β - undetermined constants depending on details of dynamics. From $\mathbf{3} \otimes \mathbf{3}^* = \mathbf{1} \oplus \mathbf{8}$, $\mathbf{3}^* \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8}$ it is seen that the operators $V_3 V_{\bar{3}}$ and $V_{\bar{3}} V_3$ from (17), (18) are just the isosinglets needed in eq.(20). As result, mass operator in (20) with redefined M_0, α, β is $\hat{M} = M_0 \mathbf{1} + \alpha V_3 V_{\bar{3}} + \beta V_{\bar{3}} V_3$, or

$$\hat{M} = M_0 \mathbf{1} + \alpha E_3^+ E_3^- q^Y + \beta E_3^- E_3^+ q^Y \quad (21)$$

where $Y = (H_1 + 2H_2)/3$ is hypercharge. Matrix elements (20) with \hat{M} from (21) are evaluated by embedding **8** in a particular representation of $U_q(su_4)$. Say, if one takes the adjoint **15** of $U_q(su_4)$, the evaluation of baryon masses yields: $M_N = M_0 + \beta q$, $M_\Sigma = M_0$, $M_\Lambda = M_0 + \frac{[2]}{[3]}(\alpha + \beta)$, $M_\Xi = M_0 + \alpha q^{-1}$. Excluding M_0, α and β , we finally obtain

$$[3]M_\Lambda + M_\Sigma = [2](q^{-1}M_N + qM_\Xi). \quad (22)$$

This alternative q -analog of octet mass relation looks much simpler than the former q -analog (9). This same q -relation (22) results from embedding **8** in any other admissible dynamical representation. What concerns empirical validity [11] of (22), there is no other way to fix the q -parameter as by usual fitting (for each of the values $q_{1,2} = \pm 1.035$, $q_{3,4} = \pm 0.903\sqrt{-1}$, the q -MR (22) indeed holds within experimental uncertainty). This is in sharp contrast with the q -analogs (9) for which there exists an appealing possibility to fix q in a rigid way by zeros of relevant polynomial A_q .

Summarizing we should stress that, although the use of Hopf-algebra structure leads to simple and mathematically appealing result eq.(22), from the physical (phenomenological) viewpoint the version (9) of q -analog obtained by applying only the tools of representation theory of quantum algebras and not strictly q -covariant symmetry breaking part in mass operator, provides much more interesting results. Among these is the degeneracy lifting and the possibility to choose among a variety of dynamical representations, defining polynomials and, thus, within discrete set of viable mass sum rules. That led us to the best MSR (11) (or (13)) for octet baryons.

8. On the connection: deformation parameter \leftrightarrow Cabibbo angle

In 3-flavor case of vector mesons, the deformation angle $\frac{\pi}{5}$ that determines ϕ -meson in (3) coincides remarkably with ω - ϕ mixing angle (known [11] to be $\theta_{\omega\phi} = 36^\circ$) of traditional $SU(3)$ -based scheme. In other words, the concept of q -deformed flavor symmetries *is closely related* with the issue of singlet mixing.

For pseudoscalar (PS) mesons, the generalization [19] of GMO-formula

$$f_\pi^2 m_\pi^2 + 3f_\eta^2 m_\eta^2 = 4f_K^2 m_K^2 \quad \text{with} \quad 1/f_\pi^2 + 3/f_\eta^2 = 4/f_K^2, \quad (23)$$

involves decay constants as coefficients. Presented in the equivalent form³

$$m_\pi^2 + \frac{9f_K^2/f_\pi^2}{4 - f_K^2/f_\pi^2} m_\eta^2 = 4 \frac{f_K^2}{f_\pi^2} m_K^2, \quad (24)$$

³ Note that having used the additional constraint in (23) we are led to the single dimensionless quantity $\frac{f_K}{f_\pi}$ involved in the multipliers of masses.

it is to be compared with our *q-analog* (2) of GMO rewritten for PS mesons (with masses squared), namely

$$m_\pi^2 + \frac{[3]}{2[2] - [3]} m_{\eta_8}^2 = \frac{2[2]}{2[2] - [3]} m_K^2. \quad (25)$$

Without singlet mixing, it is satisfied for (the mass of) *physical* η -meson put instead of η_8 at *properly fixed* $q = q_{\text{PS}}$, and just this is meant below.

The two generalizations (24) resp. (25) yield the standard GMO mass formula in the corresponding limit of single parameter, $\frac{f_K}{f_\pi} \rightarrow 1$ resp. $q \rightarrow 1$. Moreover, the following identification is valid:

$$\frac{f_K^2}{f_\pi^2} \longleftrightarrow \frac{\frac{1}{2}[2]}{2[2] - [3]}, \quad \frac{3f_K^2/f_\pi^2}{4 - f_K^2/f_\pi^2} \longleftrightarrow \frac{\frac{1}{3}[3]}{2[2] - [3]}, \quad (26)$$

from which, using $[3]_q = [2]_q^2 - 1$, we get

$$[2]_\pm = 1 - \xi_{\pi,K} \pm \sqrt{(1 - \xi_{\pi,K})^2 + 1}, \quad \xi_{\pi,K} \equiv (4f_K^2/f_\pi^2)^{-1}. \quad (27)$$

The ratio f_K/f_π is related to the Cabibbo angle. This is evident either from the formula: $\tan^2 \theta_C = \frac{m_\pi^2}{m_K^2} \left[\frac{f_K}{f_\pi} - \frac{m_\pi^2}{m_K^2} \right]^{-1}$ (see [20]), or from the formula

$$\frac{\Gamma_{K \rightarrow \mu\nu}}{\Gamma_{\pi \rightarrow \mu\nu}} = (\tan \theta_C)^2 \frac{f_K^2}{f_\pi^2} \frac{M_K}{M_\pi} \left(\frac{1 - (M_\mu/M_K)^2}{1 - (M_\mu/M_\pi)^2} \right)^2$$

for the ratio of weak decay rates usually applied to determine [21, 11] f_K/f_π in terms of the Cabibbo angle, with known empirical data on decay rates and masses. Thus, the value of f_K/f_π is expressible through θ_C . Together with (26), (27) this implies: within our scheme, the (realistic value q_{PS} of) *deformation parameter is directly connected with the Cabibbo angle*.

Similar conclusion can be arrived at in another, more general context. In [22], the q -deformed lagrangian for gauge fields of the Weinberg - Salam (WS) model invariant under the quantum-group valued gauge transformations was constructed. The obtained formula [22]

$$F_{\mu\nu}^0 = \text{Tr}_q(F_{\mu\nu}) [2(q^2 + q^{-2})]^{-1/2} = B_{\mu\nu} \cos \theta + F_{\mu\nu}^3 \sin \theta, \quad (28)$$

$$F_{\mu\nu}^3 = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3 + ie^{ab3} (A_\mu^a A_\nu^b - A_\nu^a A_\mu^b) + [A_\mu^3, B_\nu] - [A_\nu^3, B_\mu],$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] + [A_\mu^a, A_\nu^a]$$

where

$$\tan \theta = (1 - q^2)/(1 + q^2), \quad (29)$$

exhibits a mixing of the $U(1)$ -component B_μ with nonabelian components A_μ^a (the third one). Introducing the new potentials $\tilde{A}_\mu = B_\mu \cos \theta + A_\mu^3 \sin \theta$, $Z_\mu = -B_\mu \sin \theta + A_\mu^3 \cos \theta$ yields nothing but definition of physical photon \tilde{A}_μ and Z -boson of WS model, where θ coincides with the Weinberg angle, $\theta = \theta_W$. Since at $\theta = 0$ the potentials B_μ and A_μ^3 get completely unmixed whereas nonzero θ (i.e., nontrivial q -deformation) provides proper mixing as a characteristic feature of the WS model, it is thus seen that the *weak mixing is adequately modelled by the q -deformation*. Moreover, formula analogous to (29), i.e., $\tan \theta_W = q\sqrt{[4]/([2][3])} [1/2] [3/2]$, was obtained [23] within somewhat different approach to q -deforming the standard model.

Hence, the q -deformation realizes proper mixing in the sector of gauge fields, thus providing explicit connection between the weak angle and the deformation parameter q .

On the other hand, the relation found in [24], namely

$$\theta_W = 2(\theta_{12} + \theta_{23} + \theta_{13}), \quad (30)$$

connects θ_W with the Cabibbo angle $\theta_{12} \equiv \theta_C$ (and two other Kobayashi-Maskawa angles θ_{13}, θ_{23} ; as we deal with two lightest families, we have to discard θ_{13}, θ_{23}). The importance of (30) consists in that it links two apparently different mixings: one involved in *bosonic* (interaction) sector, the other in *fermionic* (matter) sector of the electroweak standard model.

Combining (29) and (30) (θ_{23}, θ_{13} omitted) we conclude: the Cabibbo angle *should be connected* with the q -parameter of a quantum-group (or quantum-algebra) based structure applied in the fermion sector.

It remains to recall that all our treatment in secs.4-7 using the q -algebras $U_q(su_n)$ concerned just the fermion sector although at the level of baryons as 3-quark bound states of fundamental fermions. Hence, it is natural to assert that there exists direct connection of the q -parameter involved in (13), (14) with fermion mixing angle. Setting $\theta_{10} = g(\theta_C)$ and $\theta_8 = h(\theta_C)$ we find for the functions $g(\theta_C)$ and $h(\theta_C)$ remarkably simple explicit form:

$$\theta_{10} = \theta_C, \quad \theta_8 = 2 \theta_C. \quad (31)$$

With $\theta_8 = \frac{\pi}{7}$ (see (11)) this suggests for Cabibbo angle the exact value $\frac{\pi}{14}$.

9. Discussion

Quantum groups and their Hopf dual counterpart - quantum universal enveloping algebras (QUEA) incorporate transformation/covariance properties of related quantum vector spaces [25]. In the context of quantum homogeneous spaces (see e.g., [26]) the corresponding quantum groups act (say, on their noncommuting 'coordinates') in a nonlinear way, as it was

exemplified [27] with quantum CP_q^n . Both quantum groups and their dual QUEA provide necessary tools in constructing [28, 17] covariant differential calculi and particular noncommutative geometry on quantum spaces.

In the case at hand the *internal* symmetries, underlying our treatment of baryon mass sum rules in secs. 4-7 and based on the broken $U_q(su_n)$ ($n \geq 3$) as well as unbroken isospin $U_q(su_2)$ q -algebras, are closely related to certain internal or extra (as regards the Minkowski space $M^{3,1}$) spacetime dimensions. From this we infer the following. The above justified direct link (31) between the Cabibbo angle $\theta_C = \frac{\pi}{14}$ and the q -parameter, which measures strength of q -deformation for the q -algebras $U_q(su_n)$ of flavor symmetry, can be viewed as an indication of noncommutative-geometric origin of fermion mixing. In this context, the value $\theta_C = \frac{\pi}{14}$ of the Cabibbo angle would serve as the noncommutativity measure of relevant quantum space (responsible for the mixing and explicitly as yet unknown) in extra dimensions. Concerning the latter, one can assert that their number is not less than 2.

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